UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
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ECE 204 *Numerical methods*

Approximating solutions to systems of higher-order initial-value problems

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Introduction

- In this topic, we will
  - Discuss converting systems of higher-order initial-value problems into a system of 1st-order initial-value problems
  - Look at an example

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System of two 2nd-order initial-value problems

- We will show this by example:

$$y^{(2)}(t)+2y^{(1)}(t)+y(t)+z(t)=\sin(t)$$
$$z^{(2)}(t)+z^{(1)}(t)+z(t)+y(t)=\cos(t)$$

  – This requires four initial conditions:

$$y(t_0)=y_0$$
$$y^{(1)}(t_0)=y_0^{(1)}$$
$$z(t_0)=z_0$$
$$z^{(1)}(t_0)=z_0^{(1)}$$

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System of two 2nd-order initial-value problems

- We will represent:

$$y(t)=w_0(t)$$
$$y^{(1)}(t)=w_1(t)$$
$$z(t)=w_2(t)$$
$$z^{(1)}(t)=w_3(t)$$

  – We can immediately translate the initial conditions:

$$y(t_0)=y_0 \qquad w_0(t_0)=y_0$$
$$y^{(1)}(t_0)=y_0^{(1)} \qquad w_1(t_0)=y_0^{(1)}$$
$$z(t_0)=z_0 \qquad w_2(t_0)=z_0$$
$$z^{(1)}(t_0)=z_0^{(1)} \qquad w_3(t_0)=z_0^{(1)}$$

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System of two 2nd-order initial-value problems

- We also immediately note that:

$$w_0^{(1)}(t) = y^{(1)}(t) = w_1(t)$$
$$w_2^{(1)}(t) = z^{(1)}(t) = w_3(t)$$

- That leaves us with determining $w_1^{(1)}(t), w_3^{(1)}(t)$

$$y^{(2)}(t) = \sin(t) - 2y^{(1)}(t) - y(t) - z(t)$$
$$z^{(2)}(t) = \cos(t) - z^{(1)}(t) - z(t) - y(t)$$

$$w_1^{(1)}(t) = \sin(t) - 2w_1(t) - w_0(t) - w_2(t)$$
$$w_3^{(1)}(t) = \cos(t) - w_3(t) - w_2(t) - w_0(t)$$

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- Thus, we have:

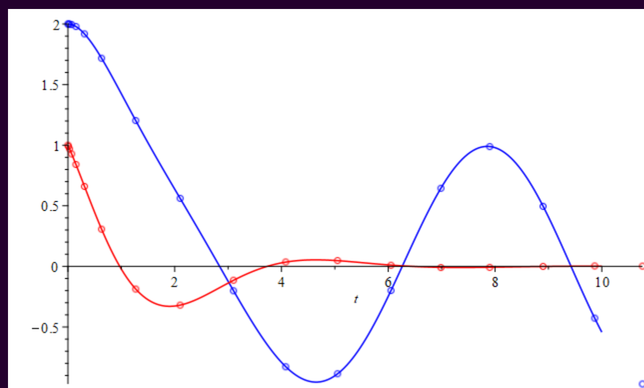
$$\mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ \sin(t) - 2w_1(t) - w_0(t) - w_2(t) \\ w_3(t) \\ \cos(t) - w_3(t) - w_2(t) - w_0(t) \end{pmatrix} \qquad \mathbf{w}(t_0) = \begin{pmatrix} y_0 \\ y_0^{(1)} \\ z_0 \\ z_0^{(1)} \end{pmatrix}$$

- We can write this as a function:

```
vector f( double t, vector w ) {
    return vector{ 4, (double[]){
        w(1),
        std::sin(t) - 2*w(1) - w(0) - w(2),
        w(3),
        std::cos(t) -   w(3) - w(2) - w(0)
    } };
}
```

System of two 2nd-order initial-value problems

- Plotting the actual solution versus the



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System of two 2nd-order initial-value problems

	t	$w_0(t) = y(t)$	$w_1(t) = y^{(1)}(t)$	$w_2(t) = z(t)$	$w_3(t) = z^{(1)}(t)$
	0	< 1	-1	2	0 >
	0.01	< 0.9899507	-1.0098008	1.9999005	-0.0198502>
	0.03	< 0.9695678	-1.0282224	1.9991135	-0.0586547>
	0.07	< 0.9277737	-1.0604838	1.9952704	-0.1327130>
	0.15	< 0.8408958	-1.1077690	1.9791628	-0.2669310>
	0.31	< 0.6599369	-1.1418081	1.9182831	-0.4828253>
	0.63	< 0.3072558	-1.0298129	1.7176558	-0.7357848>
	1.27	<-0.1870949	-0.4857930	1.2033141	-0.8119151>
	2.1015668	<-0.3197583	0.0966596	0.5615785	-0.7482154>
18	3.1015668	<-0.1142981	0.2232183	-0.2015183	-0.7630420>
	4.0791478	< 0.0363239	0.0711120	-0.8282826	-0.4258318>
	5.0504978	< 0.0478470	-0.0300383	-0.8858265	0.3398003>
	6.0504978	< 0.0094451	-0.0317667	-0.1990892	0.9286665>
	6.9858539	<-0.0096116	-0.0052132	0.6446146	0.7403237>
	7.9002755	<-0.0081134	0.0075244	0.9883434	-0.0443953>
	8.9002755	<-0.0001790	0.0047968	0.4956264	-0.8559776>
	9.8671404	< 0.0035653	-0.0015659	-0.4269945	-0.8992590>
	10.762308	< 0.0023111	-0.0031120	-0.9703157	-0.2322476>

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Other systems higher-order initial-value problems

- Thus, if we had four coupled ODEs:

$$u_1^{(4)}(t) = f_1\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_2^{(2)}(t) = f_2\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_3^{(2)}(t) = f_3\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_4^{(3)}(t) = f_4\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$

  – This requires eleven initial conditions
  – This would require us to define a system of eleven 1st-order initial-value problems

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Other systems higher-order initial-value problems

- Thus we would reformulate as follows:

$$u_1^{(4)}(t) = f_1\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_2^{(2)}(t) = f_2\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_3^{(2)}(t) = f_3\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$
$$u_4^{(3)}(t) = f_4\left(t, u_1(t), \ldots, u_4^{(2)}(t)\right)$$

$$w_0^{(1)}(t) = w_1(t)$$
$$w_1^{(1)}(t) = w_2(t)$$
$$w_2^{(1)}(t) = w_3(t)$$
$$w_3^{(1)}(t) = f_1(t, \mathbf{w}(t))$$
$$w_4^{(1)}(t) = w_5(t)$$
$$w_5^{(1)}(t) = f_2(t, \mathbf{w}(t))$$
$$w_6^{(1)}(t) = w_7(t)$$
$$w_7^{(1)}(t) = f_3(t, \mathbf{w}(t))$$
$$w_8^{(1)}(t) = w_9(t)$$
$$w_9^{(1)}(t) = w_{10}(t)$$
$$w_{10}^{(1)}(t) = f_4(t, \mathbf{w}(t))$$

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Summary

- Following this topic, you now
  - Understand how to convert a system of higher-order initial-value problems into a system of 1st-order initial-value problems
  - Have seen an example and its solution

References

[1] https://en.wikipedia.org/wiki/Initial_value_problem

Acknowledgments

None so far.

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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

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